APPLIED MATHEMATICS IN THE CLASSROOM[1]

Heather Dallas and David Mumford

Most of the mathematics students currently encounter is *merely* abstract and if connected to the world in which they live, the connection is weak. This disconnect between school mathematics and the real world leaves many students unprepared to solve many of the mathematical problems they encounter in their adult lives and convinced that these encounters will rarely occur. Instead, this article calls for K-12 mathematics classrooms that place equal emphasis of *applied and pure* mathematics, and provides a series of examples illustrating how this might be accomplished. Further, this article asserts that this *balance* of pure and applied topics should be most often *sequenced*, with new mathematical concepts introduced first in an applied setting upon which abstract understandings can be built. We hope the article will stimulate an open discussion around what types of middle and high school mathematics experiences best address the needs of all our students, not just those going on to STEM careers. We argue not for dropping abstract ideas from the curriculum but rather for re-examining the *balance* and *sequencing* of pure and applied mathematical topics in the K-12 classroom.

Our proposals are not made without some understanding of the lay of the land. There are reasons for the current emphasis on abstract mathematics in K-12 mathematics classrooms. For one, the past decade has seen too many teachers pressured by pacing plans and assessments to drag students at 80 mph through standards focused on fluency with abstract procedures. In addition, the mathematical training of teachers has often failed to

[1] This article is an expanded version of a talk given by DM at The Curtis Center for Mathematics and Teaching, UCLA, on March 2, 2013.

provide sufficient experiences with applied mathematics, and so teachers have not been prepared to engage students in real world contexts and problems. Nevertheless, the CCSS-M high school conceptual category, *mathematical modeling*, provides an opportunity for teachers to integrate real life and abstract concepts, and we argue that combining pure and applied mathematical topics best engages and prepares all our students for 21st century life. We argue for this with a small selection of examples from a vast set of applied contexts from which to draw when teaching mathematics to the 21st century student.

Financial Contexts

Financial problems provide real world "bridging contexts" (Kalchman and Koedinger 2005, 171) which can motivate and develop student understanding of the concepts and usefulness of formal algebra and function. In addition, all students will need to make multiple financial decisions important to their well-being. Yet a large percentage is bewildered by "the math" and relies on professionals – who may not act in their best interest – to make these decisions. All high school students should be given opportunities to access the mathematics necessary to make personal financial decisions that are in their best interest.

At City High School, mathematics teacher, Mrs. Bailey had students examine the result of incurring a debt with one of the many payday cash advance businesses in the local neighborhood. These local businesses provide cash two weeks in advance of a coming paycheck for a "small finance charge", which is a fixed percent of the amount borrowed. Should the lender need more than two weeks to pay off the loan, an additional finance charge is accrued, at the same percentage, but of the total now due. For instance, one company charges $15 for borrowing $100 for the first two-week period, a 15% finance charge. So after two weeks, the borrower owes \$115. If unable to pay for another two weeks, the borrower owes $\$115(1.15) = \$100(1.15)^2 = \$132.25$ (CCSS-M F-LE 1c).

One key tool that students can use to further examine these types of financial scenarios is the spreadsheet (e.g. Excel). (CCSS-M Mathematical Practice 5) Students can use any cell to record a variable such as the number of two-week periods needed to pay back the loan. For instance, the cell at column C, row 10, is referred to as cell C10. A student can enter into cell C10, “1” to indicate borrowing money for 1 two week period. They can then type in cell D10:

$$\boxed{=100(1.15)^\wedge C10}$$

What appears on the spreadsheet in D10 then is not the equation but \$115, the amount needed to pay off the loan. And if the student changes the number in C10 to 2 (for 2 two week periods) the product in D10 changes to \$132.25. Students will find \$3,785.68 appear when they change the value in C10 to 26, the number of two week periods in a year! Bearers of a U.S. high school diploma should possess the mathematical skills to avoid these payday cash advance traps.

Teachers can also use car payments as bridging contexts for polynomials. The payments on a car loan are calculated so that a constant monthly payment decreases the outstanding balance while also paying interest on this declining balance. At first, most of the payment goes into interest and near the end of the life of the loan, almost all the payment goes to pay off the balance. To begin, students can choose reasonable numerical values for their loan amount, their monthly interest rate, and for the number of months they plan to take to pay off the loan. Students can then figure out the resulting necessary monthly payment.

After investigating specific cases, students can then look toward a general case, using pay to represent the monthly payment, loan for the original loan amount and r for the *monthly* interest rate ($1/12^{th}$ the annual rate). After one month, interest increases the loan but the end of month payment decreases it, so that the balance is given by loan *(1+r) - pay. To represent the balance after two payments, replace loan in the previous expression by the expression itself, resulting in the following quadratic polynomial in 1+r:

$((\text{loan } (1+r) - \text{pay}) * (1+r) - \text{pay})$

$= \text{loan } (1+r)(1+r) - \text{pay} * (1+r) - \text{pay}$

$= \text{loan } (1+r)^2 - \text{pay} * (1+r)^1 - \text{pay}$

They can similarly represent their balance after three payments as:

$((\text{loan } (1+r) - \text{pay}) * (1+r) - \text{pay})(1+r) - \text{pay}$

$= \text{loan } (1+r)^3 – \text{pay } (1+r)^2 – \text{pay } (1+r)^1 - \text{pay } (1+r)^0$

Students can then work out what the monthly payment will be if they plan to pay off the loan after three months:

$0 = \text{loan} * (1+r)^3 - \text{pay} * (1+r)^2 - \text{pay} * (1+r)^1 - \text{pay} * (1+r)^0$

$\text{pay} * (1+r)^2 + \text{pay} * (1+r)^1 + \text{pay} * (1+r)^0 = \text{loan} * (1+r)^3$

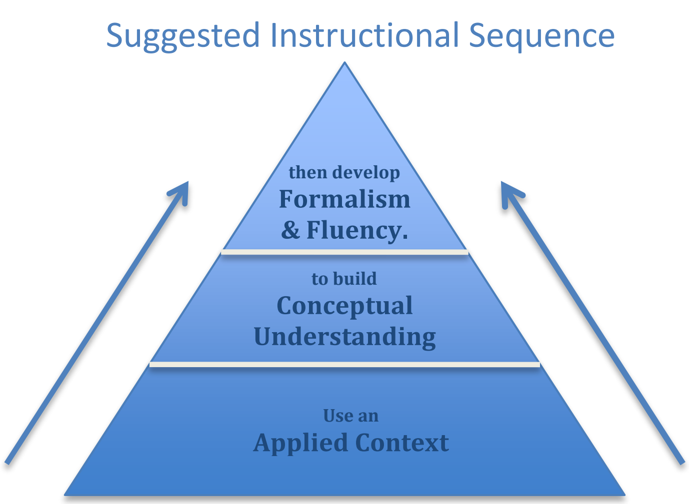
$\text{pay} * [(1+r)^2 + (1+r) + 1] = \text{loan} * (1+r)^3$

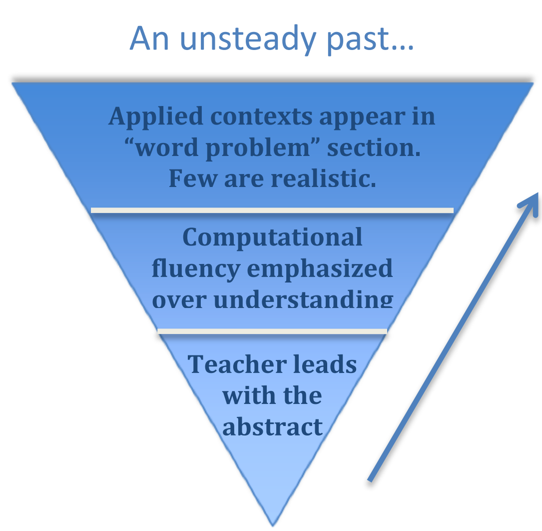
$\text{pay} = \text{loan} * (1+r)^3 / [(1+r)^2 + (1+r) + 1]$ (CCSS-M A-SSE 2 and CCSS-M A-CED 1)

Of course, most people do not pay off a car loan in only three months, and so students can work out what the monthly payment will be if they plan to pay off the loan over three years, or 36 months. This calculation nicely creates a need then for the formula for the partial sum of a geometric series as students must compute $\sum_{k=0}^{35}(1+r)^k$ (CCSS-M A-

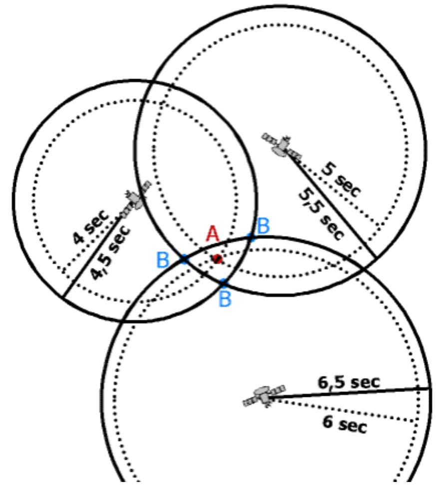
SSE 4). Students can then take the time to derive the formula for the partial sum of a geometric series, facilitated by guiding questions from their instructor. This is an instance of Guershon Harel's Necessity Principle: *Students are most likely to learn when they see a need for what we intend to teach them* (Harel, 2007, 13).

In these two examples, we see that by using the applied context of neighborhood cash advance businesses, a teacher can introduce an abstract mathematical object, the function, in particular the exponential function. By using the example of the car payment, a teacher can introduce abstract mathematical objects such as geometric series and partial sums of series. In addition, the car payment example can demonstrate to students how a real world problem can lead to a more general mathematical question, motivating the derivation of a general formula, a special case of which then solves the initial problem. The car payment context helps orient the traditional curriculum (polynomials, equation solving, and geometric series) to the needs of the majority and the real world context helps students understand the abstraction to follow. These ideas are expressed in the following diagrams:





The Mathematics of GPS



A greater emphasis on applied mathematics should also be considered in geometry. After all, the word "geometry" means "measuring the world" and this is how the subject began some four millennia ago. One modern example is measuring a person's location via GPS. The signals from three satellites are needed to get one's position assuming you are at sea level (four if not). In the figure to the right, everything is made two dimensional for clarity. The three satellites emit pulses at exactly known times – in this case we assume that all three satellites emit pulses at the same time. The circles show how far the three pulses, travelling at the speed of light, reach at two different times, each circle being labelled by the time when the corresponding signal reaches it. At point *A*, the top left signal arrives with a delay of 4 seconds, the top right with a delay of 5 seconds and the bottom (furthest) satellite is heard after 6 seconds. If your clock was perfectly synchronized with the satellites' clocks, you could measure your distance from each satellite by noting when each pulse reached you. But this is not practical. Nonetheless you can use your clock to measure the delays between the arrival times of the pulses from the three satellites.

If you are at point *A* in the figure, the delays are 1 and 2 seconds. The point is that there is only one place on earth where the three signals would be detected with exactly these delays. In the figure each of the points *B* records one pair with the same delay but not all three. The delays tell you the difference between your distances from the corresponding pair of satellites. Knowing where the satellites are, it becomes an interesting math problem to get your position, a problem which was studied by Apollonius and Newton. Students can solve this problem both algebraically and geometrically. Using algebra, you have three unknowns: your position *(x,y)* and your distance *r* from one satellite. Given *r* and the delays d_2, d_3 between its signal and the other two, you know your distance from the other satellites and these distances give you three quadratic equations in *x,y,* and *r*:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

$$(x-x_2)^2 + (y-y_2)^2 = (r + d_2)^2$$

$$(x - x_3)^2 + (y - y_3)^2 = (r + d_3)^2 \qquad \text{(CCSS-M A-CED 3)}$$

Subtracting the first equation from the second and third, you get two linear equations in *x,y,* and *r*, so you solve for x and y in terms of *r* and then plug this back into the first equation, getting a quadratic equation for r^2. The geometric approach is also beautiful, with students constructing the point (x,y) by straightedge and compass.

A Court Room Context

Statistics certainly need not be a dry memorization of formulas for various statistical tests. Rather, so much relevant data all around us can motivate and illustrate the learning of statistics. For instance, statistical significance can be investigated via the context of a stunning example from our court system posted on Tim Gowers blog on June 6th, 2008. The high school mathematics involved was sadly missed by the jury:

> "In 1972 Diana Sylvester was raped and killed in San Francisco. Despite one or two leads, the police failed to solve the case. However, they kept some DNA, and in 2006

[2] See http://en.wikipedia.org/wiki/Problem_of_Apollonius, section 3.3 or http://mathworld.wolfram.com/ApolloniusProblem.html for more details.

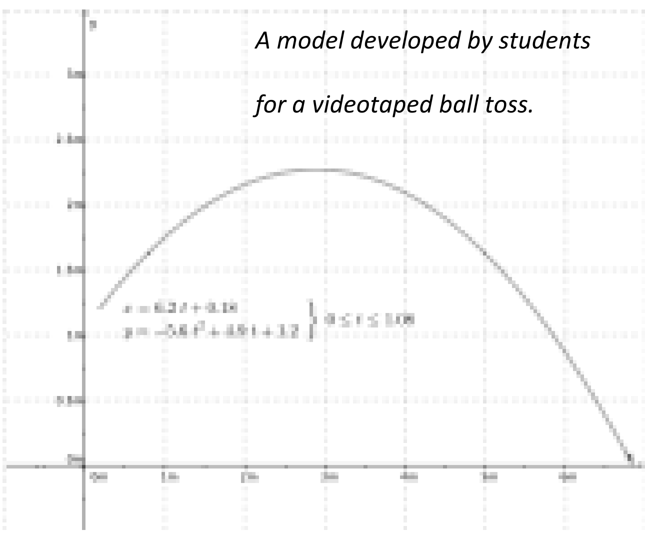
> they checked it against a DNA database of 300,000 convicted sex offenders. They discovered that it matched the DNA of John Puckett, who had spent a total of 15 years in jail for two rapes. There was no other evidence linking Puckett to the crime, but the probability that a random person's DNA would match that of the sample was judged to be 1 in 1,000,000. On that basis, he was found guilty and sentenced to life imprisonment. How reliable was the conviction?" (Gowers, 2008)

Here's the problem: suppose none of the 300,000 convicted sex offenders was the actual perpetrator. If that were the case, what is the chance of none of them matching by accident? Each one might match by accident with probability 0.000001 so would not match with probability 0.999999. The probability of none of them matching is then $(0.999999)^{300000}$ which is roughly 0.74 (CCSS-M S-CP 5). In other words, even if all of them are innocent, the chance of one of them coming up as a match is still approximately 26%. So the odds of making an error by convicting an innocent man now turns out not to be one in a million but merely one in four! Perhaps the jury would still convict him but it is not an open and shut case. Not appreciating statistical significance here was a matter of life and death and underscores how important it is to teach students when to be skeptical of probabilities (as in Disraeli's quip – 'lies, damned lies and statistics').

Applied Contexts in Physics

An especial mistake, we believe, in high school curricula is separating math and physics. Physics is most clearly expressed in the language of mathematics and mathematics comes alive from the models that arise in elementary physical experiments. Topics like the parabolic trajectory of a ball thrown in the air, are traditional fare as "aftermath" in the "word problem" section of secondary mathematics texts. But these contexts can be brought alive in the classroom to help students make sense of the related mathematics.

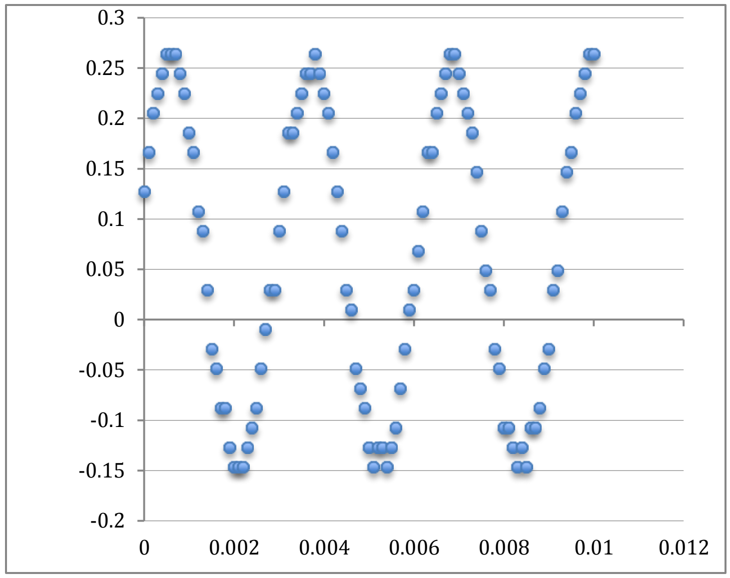
For instance, a teacher can use projectile motion to help students make sense of quadratic functions. If ten or so students stand at regular intervals along the length of a wall, a ball can be tossed between the students and the wall, and students can place a marker on the wall to indicate the height of the ball as it passed. In this manner, vertical and horizontal measurements can be gathered, to estimate the path of the ball in flight. Students can graph the data using a graphing utility, and choose values of *a*, *b*, and *c* to develop an equation of the form $y = a(x-b)^2 + c$ to model the path. As the students work, the trail of markers on the classroom wall can serve as a concrete reminder of the real world context represented by the points on the graph and the developed equation. Students can use their model in various ways, including estimating where the ball hit the ground (CCSS-M A-REI 4).



A model developed by students for a videotaped ball toss.

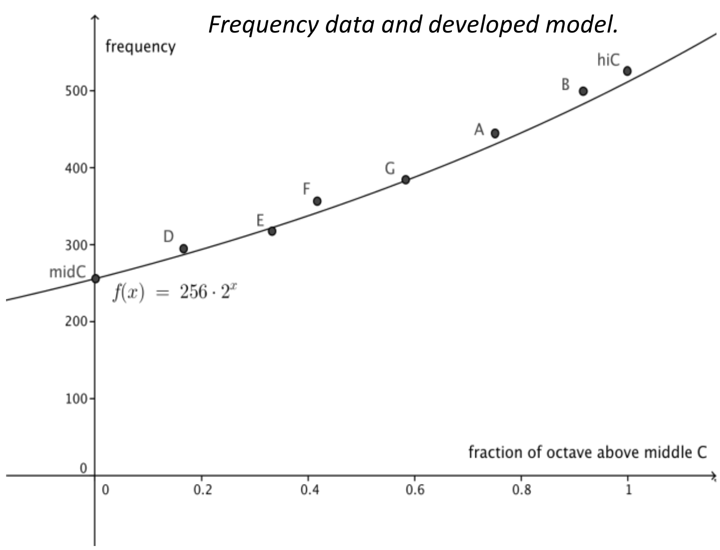
The ball toss can also be videotaped, and the tape used to determine the time at which the ball passed each student. Thus students can additionally investigate how the ball's horizontal and vertical position each vary with time. Students can relate the coefficient of the linear equation to the motion (an estimate of the horizontal velocity of the ball) or compare the quadratic model they developed for the path to the quadratic model they developed to describe how the vertical position varied with time (CCSS-M F-LE 5).

A less standard physics context is sound. The connection of the math of periodic functions to the physics of pressure waves in air and thus to music is wonderful. Students can use an oscilloscope, a Venier microphone probe, or a computer app to gather data relate to the pressure wave generated by any sound. Students can then develop a sinusoidal model for the wave.



Data gathered from an E tuning fork: displacement (mm) vs. time (sec)

In addition, students can use the data to estimate the frequency of the note (the reciprocal of period). If this is done for each of the eight notes in the scale, the class can graph estimated frequencies for the whole major scale as shown in the figure. They can develop a model for the relationship between a note's frequency (in Hz) and the number of octaves it is above middle C (CCSS-M F-LE 1c and F-LE 5). In this case, it is true that the frequencies of notes double from octave to octave. Students can use their model to estimate at how many octaves above middle C a particular frequency is reached (CCSS-M F-LE 4). They can also



compare how much moving up a step in the middle octave on a piano effects the frequency of a note, compared to the effect of the same move one octave higher (CCSS-M F-IF 6).

The contexts of projectile motion and sound motivate computation. Wanting to know what number of octaves above middle C is associated with a particular frequency provides a *need* for logarithms. Similarly, wanting to know where and when the ball hits the ground provides a *need* to solve a quadratic and a linear equation. And again, the applied context provides a concrete foundation on which to build student understanding of the abstract mathematics in addition to providing a *necessity* for, or motivating it.

Upon reading this, you might be concerned that these discussions short-change pure mathematics. Consider a response to a Huffington Post piece written by the senior author and Sol Garfunkel:

> "You do not study mathematics because it helps you build a bridge. You study mathematics because it is the poetry of the universe. Its beauty transcends mere *things."* Jonathan Farley (Prof. of Computer Science, University of Maine)

But our point is that you can do *both* pure and applied math, something that current K-12 mathematics students largely do not. Archimedes not only proved some of the most beautiful theorems, but built armaments to hold off the Roman siege of Syracuse. A gem in analysis – Fourier series – was invented to describe the motion of a vibrating string, the motion of the moon and the underground cooling of the soil in winter. *Pure and applied math are inseparable Siamese twins.*

In fact, "There is a healthy continuum between research in the mathematical sciences, which may or may not be pursued with an application in mind, and the range of applications to which mathematical science advances contribute. To function well in a technologically advanced society, every educated person should be familiar with multiple aspects of the mathematical sciences.” (Committee on the Mathematical Sciences in 2025, 2014, 1) And while this article only presents a few examples of this continuum, we hope it provides enough to inspire others to rethink what mathematical experiences truly help the majority of our students acquire the mathematical understandings and ways of thinking they will need for a rewarding adult life. There will always be disagreements, but such a discussion seems to be the right basis on which to plan something so important as our children’s future.

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